

Zero Energy Bound States and Resonances in Three-Particle Systems.

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Abstract. We consider a three-particle system in \mathbb{R}^3 with non-positive pair-potentials and non-negative essential spectrum. Under certain restrictions on potentials it is proved that the eigenvalues are absorbed at zero energy threshold given that there is no negative energy bound states and zero energy resonances in particle pairs. It is shown that the condition on the absence of zero energy resonances in particle pairs is essential. Namely, we prove that if at least one pair of particles has a zero energy resonance then a square integrable zero energy ground state of three particles does not exist. It is also proved that one can tune the coupling constants of pair potentials so that for any given $R, \epsilon > 0$: (a) the bottom of the essential spectrum is at zero; (b) there is a negative energy ground state $\psi(\xi)$ such that $\int |\psi(\xi)|^2 d^6 \xi = 1$ and $\int_{|\xi| \leq R} |\psi(\xi)|^2 d^6 \xi < \epsilon$.

PACS numbers: 03.65.Ge, 03.65.Db, 21.45.-v, 67.85.-d, 02.30.Tb

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1. Introduction

Let us consider an N -particle Schrödinger operator

$$H(\lambda) = H_0 - \lambda \sum_{1 \leq i < j \leq N} V_{ij}(r_i - r_j), \quad (1)$$

where $\lambda > 0$ is a coupling constant, H_0 is a kinetic energy operator with the centre of mass removed, $r_i \in \mathbb{R}^3$ are particle position vectors, the pair potentials are real (further restrictions on the potentials would be given later). Suppose that for λ in the vicinity of some $\lambda_{cr} < \infty$ there is a bound state $\psi(\lambda) \in D(H_0)$ with the energy $E(\lambda) < \inf \sigma_{ess}(H(\lambda))$ and $E(\lambda) \rightarrow \inf \sigma_{ess}(H(\lambda_{cr}))$ when $\lambda \rightarrow \lambda_{cr}$. The question whether $E(\lambda_{cr})$ is an eigenvalue of $H(\lambda_{cr})$ was considered in various contexts in [1, 2, 3, 4, 5, 6] (the list of references is by far incomplete).

In [7], Theorem 3.3, it was claimed that if $V_{ij} \in C_0^\infty(\mathbb{R}^3)$, $V_{ij} \geq 0$, and none of the subsystems has negative energy bound states or zero energy resonances, then there exists $\psi(\lambda_{cr}) \in D(H) = D(H_0)$, $\psi(\lambda_{cr}) \neq 0$ such that $H(\lambda_{cr})\psi(\lambda_{cr}) = 0$. Unfortunately, the proof in [7] contains a mistake. In Eq. 53 of [7] the mixed term containing first order derivatives is erroneously omitted, which makes the results of Ref. 35 in [7] concerning the fall off of the wave function inapplicable. And it is not immediately clear how this mistake can be corrected. Here we prove the result stated by Karner for $N = 3$ with a different method and for a larger class of potentials (Theorem 2 of this paper).

Naturally, a question can be raised whether the condition on the absence of 2-particle zero energy resonances is essential. Here we show that indeed it is. Namely, in Sec. 5 (Theorem 3) we prove that the 3-particle ground state at zero energy can be at most a resonance and not a L^2 state if at least one pair of particles has a resonance at zero energy. The method of proof is inspired by [1, 8, 9]. The last section provides an example of a 3-particle system, where each 2-particle subsystem is unbound, one 2-particle subsystem is at the 2-particle coupling constant threshold and the whole 3-particle system has a resonance but not a bound state at zero energy. Systems like this can always be constructed through appropriate tuning of the coupling constants.

Note, that the 3-particle case differs essentially from the 2-particle case, where under similar restrictions on pair potentials the zero energy ground state cannot be a bound state [1, 6]. This difference has far reaching physical consequences, which concern the size of a system in its ground state (we ignore the particle statistics here). In the two-particle case the size of a system in the ground state can be made infinite by tuning, for example, the coupling constant so that the bound state with negative energy approaches the zero energy threshold [6]. In the three particle case the size of the system remains finite, given that in the course of tuning the coupling constants of two-particle subsystems stay away from critical values, at which the two-particle zero energy resonances appear. To underline the connection with the size of the system we formulate the proofs in terms of spreading and non-spreading sequences of bound states. The obtained results are relevant in the physics of halo nuclei [10], molecular physics [11] and Efimov states [12]. Here we refer the reader to the last section, where we discuss

possible physical applications of our results.

The paper is organized as follows. In Sec. 2 we use the ideas of Zhislin [13] to set up the framework for the analysis of eigenvalue absorption in connection with the spreading of sequences of wave functions. Here we prefer to maintain generality and do not restrict ourselves to $N = 3$. In Sec. 3 we consider the 3-particle case and employ the equations of Faddeev type to prove Theorem 2, which is the main result of this section. In Sec. 4 we prove an auxiliary result concerning the two-particle zero energy resonance. In Sec. 5 we prove Theorem 3, which says that the ground state of three particles cannot be a bound state given that there is no bound states in particle pairs and at least one particle pair has a zero energy resonance. The last section provides a constructive example of a critically bound three-particle system with such conditions. In the last section we also give the overview of relevant physical phenomena and discuss possible applications of our results.

2. Spreading and Bound States at Threshold

The main result of this section (Theorem 1) appears implicitly in [13], where Zhislin considers minimizing sequences of the energy functional in Sobolev spaces. For our purposes it is more useful to consider sequences of eigenstates and use an approach in the spirit of [3].

Consider the N -particle Hamiltonian, which depends on a parameter

$$H(\lambda) = H_0 + V(\lambda), \quad (2)$$

$$V(\lambda) = \sum_{1 \leq i < j \leq N} V_{ij}(\lambda; r_i - r_j), \quad (3)$$

where H_0 is the kinetic energy operator with the centre of mass removed, V_{ij} are pair potentials and $r_i \in \mathbb{R}^3$ are position vectors. For the parameter λ we assume that $\lambda \in \mathbb{R}$ (this is done for clarity, in fact, λ can take values in a topological space). We impose the following set of restrictions.

R1 $H(\lambda)$ is defined for an infinite sequence of parameter values $\lambda_1, \lambda_2, \dots$ and λ_{cr} , where $\lim_{n \rightarrow \infty} \lambda_n = \lambda_{cr}$.

R2 $|V_{ij}(\lambda; y)| \leq F(y)$ for all λ defined in R1, where $V_{ij}, F \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$.

R3 $\forall f \in C_0^\infty(\mathbb{R}^{3N-3})$: $\lim_{\lambda_n \rightarrow \lambda_{cr}} \|[V(\lambda_n) - V(\lambda_{cr})]f\| = 0$.

The symbol L^∞ denotes bounded Borel functions going to zero at infinity. By R2 $H(\lambda)$ is self-adjoint on $D(H_0)$ [14].

The bottom of the essential spectrum of $H(\lambda)$ is denoted as

$$E_{thr}(\lambda) := \inf \sigma_{ess}(H(\lambda)). \quad (4)$$

The set of requirements on the system continues as follows

R4 for all λ_n there is $E(\lambda_n) \in \mathbb{R}, \psi(\lambda_n) \in D(H_0)$ such that $H(\lambda_n)\psi(\lambda_n) = E(\lambda_n)\psi(\lambda_n)$, where $\|\psi(\lambda_n)\| = 1$ and $E(\lambda_n) < E_{thr}(\lambda_n)$.

$$\text{R5 } \lim_{\lambda_n \rightarrow \lambda_{cr}} E(\lambda_n) = \lim_{\lambda_n \rightarrow \lambda_{cr}} E_{thr}(\lambda_n) = E_{thr}(\lambda_{cr}) .$$

The requirements R4-5 say that for each n the system has a level below the continuum and for $\lambda_n \rightarrow \lambda_{cr}$ the energy of this level approaches the bottom of the continuous spectrum.

In the proofs we shall use the term “spreading sequence”, which is due to Zhislin [13]. The sequence of functions $f_n(x) \in L^2(\mathbb{R}^d)$ **spreads** if there is $a > 0$ such that $\limsup_{n \rightarrow \infty} \|\chi_{\{|x| > R\}} f_n\| > a$ for all $R > 0$. (the notation χ_Ω always means the characteristic function of the set Ω). The sequence f_n is called **totally spreading** if $\lim_{n \rightarrow \infty} \|\chi_{\{|x| \leq R\}} f_n\| = 0$ for all $R > 0$. Note that any normalized sequence, which converges in norm, does not spread, and any sequence, which goes to zero in norm, totally spreads.

Lemma 1. *Let $H(\lambda)$ be a Hamiltonian satisfying R1-5. Then*

$$\sup_n \|H_0 \psi(\lambda_n)\| < \infty. \quad (5)$$

Proof. The statement represents a well-known fact, see e. g. [13] but for completeness we give the proof right here. The Schrödinger equation $H_0 \psi(\lambda_n) = -V(\lambda_n) \psi(\lambda_n) + E(\lambda_n) \psi(\lambda_n)$ gives the bound $\|H_0 \psi(\lambda_n)\| \leq \|V(\lambda_n) \psi(\lambda_n)\| + \mathcal{O}(1)$. It remains to show that $\|V(\lambda_n) \psi(\lambda_n)\| = \mathcal{O}(1)$. By R2 $|V_{ij}| \leq F_{ij}$, where for a shorter notation we denote $F_{ij} := F(r_i - r_j)$. Using that F_{ij} is H_0 -bounded with a relative bound 0 (see vol.2, Theorem X.16 in [14]) we obtain

$$\begin{aligned} \|V(\lambda_n) \psi(\lambda_n)\| &= \left\| \sum_{i < j} V_{ij}(\lambda_n; r_i - r_j) \psi(\lambda_n) \right\| \leq \sum_{i < j} \|F_{ij} \psi(\lambda_n)\| \\ &\leq a \|H_0 \psi(\lambda_n)\| + b \leq a \|V(\lambda_n) \psi(\lambda_n)\| + \mathcal{O}(1), \end{aligned} \quad (6)$$

where $a, b > 0$ are constants and a can be chosen as small as pleased. Setting, for example, $a = 1/2$ gives $\|V(\lambda_n) \psi(\lambda_n)\| = \mathcal{O}(1)$. \square

The following theorem illustrates the connection between non-spreading and bound states at threshold.

Theorem 1 (Zhislin). *Let $H(\lambda)$ satisfy R1-5. If the sequence $\psi(\lambda_n)$ does not totally spread then $H(\lambda_{cr})$ has a bound state at threshold $\psi_{cr} \in D(H_0)$, so that*

$$H(\lambda_{cr}) \psi_{cr} = E_{thr}(\lambda_{cr}) \psi_{cr}, \quad (7)$$

.

For the proof we need a couple of technical Lemmas.

Lemma 2. *Suppose $f_n \in D(H_0)$ is such that $\sup_n \|H_0 f_n\| < \infty$ and $f_n \xrightarrow{w} f_0$. Then (a) $f_0 \in D(H_0)$; (b) for any operator A , which is relatively H_0 -compact $\|A(f_n - f_0)\| \rightarrow 0$.*

Proof. First, let us prove that the sequence $H_0 f_n$ is weakly convergent. A proof is by contradiction. Suppose $H_0 f_n$ has two weak limit points, i.e. there exist f'_k, f''_k , which are subsequences of f_n and for which $H_0 f'_k \xrightarrow{w} \phi_1$ and $H_0 f''_k \xrightarrow{w} \phi_2$, where $\phi_{1,2} \in L^2$ and

$\phi_1 \neq \phi_2$. On one hand, because $\phi_1 \neq \phi_2$ and $D(H_0)$ is dense in L^2 there is $g \in D(H_0)$ such that $(\phi_1 - \phi_2, g) \neq 0$. On the other hand, using that $f'_k \xrightarrow{w} f_0$ and $f''_k \xrightarrow{w} f_0$ we get

$$(\phi_1 - \phi_2, g) = \lim_{k \rightarrow \infty} [(H_0(f'_k - f''_k), g)] = \lim_{k \rightarrow \infty} [((f'_k - f''_k), H_0 g)] = 0, \quad (8)$$

a contradiction. Hence, $H_0 f_n \xrightarrow{w} G$, where $G \in L^2$. $\forall f \in D(H_0)$ by self-adjointness of H_0 we obtain $(H_0 f, f_0) = \lim_{n \rightarrow \infty} (H_0 f, f_n) = (f, G)$. Thus $f_0 \in D(H_0)$ and $G = H_0 f_0$, which proves (a). To prove (b) note that $(H_0 + 1)(f_n - f_0) \xrightarrow{w} 0$. Using that compact operators acting on weakly convergent sequences make them converge in norm we get

$$A(f_n - f_0) = A(H_0 + 1)^{-1}(H_0 + 1)(f_n - f_0) \rightarrow 0, \quad (9)$$

since $A(H_0 + 1)^{-1}$ is compact by condition of the lemma. \square

Lemma 3. Suppose $f_n \in D(H_0)$ is such that $\sup_n \|H_0 f_n\| < \infty$ and $f_n \xrightarrow{w} f_0$. Then (a) if f_n does not spread then $f_n \rightarrow f_0$ in norm; (b) if f_n does not totally spread then $f_0 \neq 0$.

Proof. Let us start with (a). Because f_n does not spread it is enough to show that $\|\chi_{\{|x| \leq R\}}(f_n - f_0)\| \rightarrow 0$ for all R in norm. And this is true because $\chi_{\{|x| \leq R\}}$ is relatively H_0 -compact [14, 15] and Lemma 2 applies. To prove (b) let us assume by contradiction that $f_n \xrightarrow{w} 0$. Using the same arguments we get that $\|\chi_{\{|x| \leq R\}} f_n\| \rightarrow 0$ for all R . But this would mean that f_n totally spreads contrary to the condition of the Lemma. \square

Proof of Theorem 1. Without loosing generality we can assume that there are $a, R > 0$ such that $\|\chi_{\{|x| < R\}} \psi(\lambda_n)\| > a$ (otherwise we can pass to an appropriate subsequence, since $\psi(\lambda_n)$ does not totally spread). By the Banach-Alaoglu theorem we choose a weakly convergent subsequence so that $\psi(\lambda_{n_k}) \xrightarrow{w} \psi_{cr}$, where $\psi_{cr} \in D(H_0)$ by Lemma 2. $\psi(\lambda_{n_k})$ does not totally spread and is weakly convergent, hence, by Lemma 3(b) $\psi_{cr} \neq 0$. For any $f \in C_0^\infty$ we have

$$\begin{aligned} ([H(\lambda_{cr}) - E_{thr}(\lambda_{cr})]f, \psi_{cr}) &= \lim_{\lambda_{n_k} \rightarrow \lambda_{cr}} ([H(\lambda_{cr}) - E_{thr}(\lambda_{n_k})]f, \psi(\lambda_{n_k})) \\ &= \lim_{\lambda_{n_k} \rightarrow \lambda_{cr}} ([H(\lambda_{n_k}) - (V(\lambda_{n_k}) - V(\lambda_{cr})) - E_{thr}(\lambda_{n_k})]f, \psi(\lambda_{n_k})) \\ &= \lim_{\lambda_{n_k} \rightarrow \lambda_{cr}} \left\{ [E(\lambda_{n_k}) - E_{thr}(\lambda_{n_k})](f, \psi(\lambda_{n_k})) - ([V(\lambda_{n_k}) - V(\lambda_{cr})]f, \psi(\lambda_{n_k})) \right\} = 0, \end{aligned} \quad (10)$$

where in the last equation we have used R3, R5. Summarizing, for all $f \in C_0^\infty$ we have

$$([H(\lambda_{cr}) - E_{thr}(\lambda_{cr})]f, \psi_{cr}) = (f, [H(\lambda_{cr}) - E_{thr}(\lambda_{cr})]\psi_{cr}) = 0, \quad (11)$$

meaning that (7) holds. \square

The following Lemmas will be needed in the next Section.

Lemma 4. A uniformly norm-bounded sequence of functions $f_n \in L^2(\mathbb{R}^n)$ having a property that every weakly convergent subsequence converges also in norm does not spread.

Proof. By contradiction, let us assume that f_n spreads. Then it is possible to extract a subsequence $g_k = f_{n_k}$ with the property $\|\chi_{\{|x| \geq k\}} g_k\| > a$, where $a > 0$ is a constant. On one hand, it is easy to see that g_k has no subsequences that converge in norm. On the other hand, by the Banach-Alaoglu theorem g_k must have at least one weakly converging subsequence, which is norm-convergent by condition of the lemma. A contradiction. \square

Lemma 5. *Suppose $g \in C(\mathbb{R}^{3N-3})$ has the property that $|g| \leq 1$ and $g = 0$ if $|r_i - r_j| < \delta|x|$, where δ is a constant. Then the operator $gF(r_i - r_j)$ is relatively H_0 -compact.*

Proof. It suffices to consider the case $F \in L^2(\mathbb{R}^3)$ (the case $F \in L^\infty(\mathbb{R}^3)$ trivially follows from Lemma 7.11 in [15]). For $k = 1, 2, \dots$ we can write

$$gF_{ij}(H_0 + 1)^{-1} = \chi_{\{|x| |r_i - r_j| < k\}} gF_{ij}(H_0 + 1)^{-1} + \chi_{\{|x| |r_i - r_j| \geq k\}} gF_{ij}(H_0 + 1)^{-1}, \quad (12)$$

where again $F_{ij} := F(r_i - r_j)$. The first operator on the rhs is compact (Lemma 7.11 in [15]). We need to show that the second one goes to zero in norm when $k \rightarrow \infty$ (in this case the operator on the lhs is compact as a norm-limit of compact operators). The following integral estimate of the square of its norm is trivial

$$\|\chi_{\{|x| |r_i - r_j| \geq k\}} gF_{ij}(H_0 + 1)^{-1}\|^2 \leq \frac{1}{(4\pi)^2} \int_{|r| \geq k} d^3r |F(r)|^2 \int d^3r' \frac{e^{-2|r'|}}{|r'|^2}. \quad (13)$$

Because $F \in L^2(\mathbb{R}^3)$ the rhs goes to zero as $k \rightarrow \infty$. \square

3. Zero Energy Bound States of Three Particles

We apply the framework of Sec. 2 to the system of three particles with non-positive potentials. The case $N > 3$ and potentials taking both signs would be considered elsewhere. For simplicity we take the parameter $\lambda > 0$ as a coupling constant of the interaction (see [1, 2])

$$H(\lambda) = H_0 - \lambda V, \quad (14)$$

$$V = \sum_{1 \leq i < j \leq 3} V_{ij}(r_i - r_j). \quad (15)$$

We shall need the following additional requirements

R6 $V_{ij} \geq 0$ and $\lambda V_{ij}(y) \leq F(y)$, where $F \in L^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ and λ takes values as defined in R1.

R7 There exists $\epsilon > 0$ such that $H_0 - (\lambda + \epsilon)V_{ij} \geq 0$ for all λ defined in R1 and all pair potentials V_{ij} .

Requirement R7 means that the two-particle subsystems have no bound states with negative energy and no resonances at zero energy. This results in $E_{thr}(\lambda) = 0$. Our aim is to prove

Theorem 2. Suppose $H(\lambda)$ defined in (14)–(15) satisfies R1, R4-7. Then for $n \rightarrow \infty$ the sequence $\psi(\lambda_n)$ does not spread and there exists a bound state at threshold $\psi_{cr} \in D(H_0)$, $\|\psi_{cr}\| \neq 0$, such that $H(\lambda_{cr})\psi_{cr} = 0$.

We shall defer the proof, which boils down to the construction of Faddeev equations [16], see also [8, 9], to the end of the section. Let us introduce an analytic operator function $B_{ij}(z)$ for each pair of particles (ij) . We shall construct B_{12} and the other two operators are constructed similarly. We use Jacobi coordinates [17] $x = [\sqrt{2\mu_{12}}/\hbar](r_2 - r_1)$ and $y = [\sqrt{2M_{12}}/\hbar](r_3 - m_1/(m_1 + m_2)r_1 - m_2/(m_1 + m_2)r_2)$, where $\mu_{ij} = m_i m_j / (m_i + m_j)$ and $M_{ij} = (m_i + m_j)m_l / (m_i + m_j + m_l)$ are reduced masses (the indices i, j, l are all different). These coordinates make the kinetic energy operator take the form

$$H_0 = -\Delta_x - \Delta_y. \quad (16)$$

Let \mathcal{F}_{12} denote the partial Fourier transform in $L^2(\mathbb{R}^6)$ acting as follows

$$\hat{f}(x, p_y) = \mathcal{F}_{12}f(x, y) = \frac{1}{(2\pi)^{3/2}} \int d^3y e^{-ip_y \cdot y} f(x, y). \quad (17)$$

Then $B_{12}(z)$ is defined through

$$B_{12}(z) = 1 + z + \mathcal{F}_{12}^{-1}t(p_y)\mathcal{F}_{12}, \quad (18)$$

where

$$t(p_y) = (\sqrt{|p_y|} - 1)\chi_{\{|p_y| \leq 1\}}. \quad (19)$$

Similarly, using other Jacobi coordinates one defines $B_{ij}(z)$ and $\mathcal{F}_{ij}(z)$ for all particle pairs. Note that $B_{ij}(z)$ and $B_{ij}^{-1}(z)$ are analytic on $\text{Re } z > 0$.

Lemma 6. The operator function in $L^2(\mathbb{R}^6)$

$$\mathcal{A}_{ij}(z) = (H_0 + z^2)^{-1}V_{ij}^{1/2}B_{ij}(z) \quad (20)$$

is uniformly bounded for $z \in (0, 1]$, and strongly continuous for $z \rightarrow +0$.

Proof. We consider the case when $(ij) = (12)$, other indices are treated similarly. Instead of $\mathcal{A}_{12}(z)$ we consider $\mathcal{F}_{12}\mathcal{A}_{12}(z)\mathcal{F}_{12}^{-1}$. We take $z \in (0, 1)$ and split the operator

$$\mathcal{F}_{12}\mathcal{A}_{12}(z)\mathcal{F}_{12}^{-1} = K_1(z) + K_2(z), \quad (21)$$

where

$$K_1(z) = (-\Delta_x + p_y^2 + z^2)^{-1}V_{12}^{1/2}(\alpha x)[t(p_y) + 1], \quad (22)$$

$$K_2(z) = (-\Delta_x + p_y^2 + z^2)^{-1}V_{12}^{1/2}(\alpha x)z \quad (23)$$

are integral operators acting on $\phi(x, p_y) \in L^2(\mathbb{R}^6)$ as

$$K_1(z)\phi = \frac{1}{4\pi} \int d^3x' \frac{e^{-\sqrt{p_y^2 + z^2}|x-x'|}}{|x-x'|} V_{12}^{1/2}(\alpha x')[t(p_y) + 1]\phi(x', p_y), \quad (24)$$

$$K_2(z)\phi = \frac{z}{4\pi} \int d^3x' \frac{e^{-\sqrt{p_y^2 + z^2}|x-x'|}}{|x-x'|} V_{12}^{1/2}(\alpha x')\phi(x', p_y). \quad (25)$$

The numerical coefficient α depends on masses $\alpha := \hbar/\sqrt{2\mu_{12}}$. Applying the Cauchy-Schwarz inequality we get

$$|K_1(z)\phi|^2 \leq \int d^3x' \frac{e^{-2|p_y||x-x'|}}{|x-x'|^2} [t(p_y) + 1]^2 V_{12}(\alpha x') \times \int d^3x' |\phi(x', p_y)|^2, \quad (26)$$

$$|K_2(z)\phi|^2 \leq z^2 \int d^3x' \frac{e^{-2z|x-x'|}}{|x-x'|^2} V_{12}(\alpha x') \times \int d^3x' |\phi(x', p_y)|^2, \quad (27)$$

where we have used $z \in (0, 1]$. Integrating (26) and (27) over x leads to

$$\int d^3x |K_1(z)\phi|^2 \leq cc'c'' \left[\int d^3x' |\phi(x', p_y)|^2 \right], \quad (28)$$

$$\int d^3x |K_2(z)\phi|^2 \leq cc' \left[\int d^3x' |\phi(x', p_y)|^2 \right], \quad (29)$$

where c, c', c'' are the following finite constants

$$c = \int d^3x' V_{12}(\alpha x'), \quad (30)$$

$$c' = \int d^3x \frac{e^{-2|x|}}{|x|^2}, \quad (31)$$

$$c'' = \sup_{p_y \in \mathbb{R}^3} [t(p_y) + 1]^2 / |p_y|. \quad (32)$$

Integrating (28)–(29) over p_y gives that $K_{1,2}(z)$ is uniformly norm-bounded for $z \in (0, 1]$. The strong continuity for $z \rightarrow +0$ follows from (24)–(25) by the dominated convergence theorem. \square

For $z \in (0, \infty)$ let us define

$$\mathcal{C}_{ik;jm}(z) = V_{ik}^{1/2} (H_0 + z^2)^{-1} V_{jm}^{1/2}. \quad (33)$$

The properties of the above operator are summarized in the following

Lemma 7. *Suppose $H(\lambda)$ defined in (14)–(15) satisfies R1, R6, R7. Then (a) the operator function $\mathcal{C}_{ik;jm}(z)$ is norm-continuous for $z > 0$ and has a norm limit for $z \rightarrow +0$; (b) there exists $\delta > 0$ such that $\lambda_n \|\mathcal{C}_{ik;ik}(z)\| < 1 - \delta$ for all $z \geq 0$.*

Proof. Below we prove that $\mathcal{C}_{ik;jm}(z)$ for $z_{1,2} \in (0, \infty)$ satisfies the following continuity condition

$$\|\mathcal{C}_{ik;jm}(z_1) - \mathcal{C}_{ik;jm}(z_2)\| \leq l \sqrt{|z_1^2 - z_2^2|}, \quad (34)$$

where l is a constant. From (34) it easily follows that $\mathcal{C}_{ik;jm}(z)$ for $z \rightarrow +0$ form a Cauchy sequence. Therefore we can define the norm limit

$$\mathcal{C}_{ik;jm}(0) := \lim_{z \rightarrow +0} \mathcal{C}_{ik;jm}(z) \quad (35)$$

and $\mathcal{C}_{ik;jm}(z)$ becomes norm-continuous for $z \geq 0$. Let us first prove (34) for $\mathcal{C}_{ik;ik}(z)$. It suffices to consider $\mathcal{C}_{12;12}(z)$. Taking $0 < z_1 < z_2$ we have

$$\|\mathcal{C}_{12;12}(z_1) - \mathcal{C}_{12;12}(z_2)\| = \|K(z_1, z_2)\|, \quad (36)$$

where

$$K(z_1, z_2) := \mathcal{F}_{12} V_{12}^{1/2} \left[(H_0 + z_1^2)^{-1} - (H_0 + z_2^2)^{-1} \right] V_{12}^{1/2} \mathcal{F}_{12}^{-1}. \quad (37)$$

The integral operator $K(z_1, z_2)$ acts on $\phi(x, p_y) \in L^2(\mathbb{R}^6)$ as

$$K(z_1, z_2)\phi(x, p_y) = \int d^3x' k(z_1, z_2; x, x', p_y)\phi(x', p_y), \quad (38)$$

where

$$k(z_1, z_2; x, x', p_y) = \frac{\sqrt{V_{12}(\alpha x)V_{12}(\alpha x')}}{4\pi|x-x'|} \left\{ e^{-\sqrt{p_y^2+z_1^2}|x-x'|} - e^{-\sqrt{p_y^2+z_2^2}|x-x'|} \right\}. \quad (39)$$

Obviously,

$$\|K(z_1, z_2)\|^2 \leq \sup_{p_y \in \mathbb{R}^3} \int d^3x d^3x' |k(z_1, z_2; x, x', p_y)|^2. \quad (40)$$

Using the inequality

$$|k(z_1, z_2; x, x', p_y)| \leq \frac{\sqrt{V_{12}(\alpha x)V_{12}(\alpha x')}}{4\pi} \sqrt{z_2^2 - z_1^2} \quad (41)$$

we obtain from (40)

$$\|K(z_1, z_2)\|^2 \leq \frac{c^2}{16\pi^2} |z_2^2 - z_1^2|, \quad (42)$$

where c was defined in (30). From (42) and (36) the continuity condition (34) follows for $\mathcal{C}_{ik;ik}(z)$. It remains to prove (34) for $\mathcal{C}_{ik;jm}(z)$. For $0 < z_1 < z_2$ by the resolvent identity we have

$$(H_0 + z_1^2)^{-1} - (H_0 + z_2^2)^{-1} = (z_2^2 - z_1^2)(H_0 + z_1^2)^{-1/2}(H_0 + z_2^2)^{-1}(H_0 + z_1^2)^{-1/2} \geq 0 \quad (43)$$

Thus we can write

$$\begin{aligned} & \|\mathcal{C}_{ik;jm}(z_1) - \mathcal{C}_{ik;jm}(z_2)\| \\ &= \left\| V_{ik}^{1/2} [(H_0 + z_1^2)^{-1} - (H_0 + z_2^2)^{-1}]^{1/2} [(H_0 + z_1^2)^{-1} - (H_0 + z_2^2)^{-1}]^{1/2} V_{jm}^{1/2} \right\| \\ &\leq \|\mathcal{C}_{ik;ik}(z_1) - \mathcal{C}_{ik;ik}(z_2)\|^{1/2} \|\mathcal{C}_{jm;jm}(z_1) - \mathcal{C}_{jm;jm}(z_2)\|^{1/2} \leq l \sqrt{|z_1^2 - z_2^2|}, \end{aligned} \quad (44)$$

where we have used that $\|AB\| \leq \|AA^\dagger\|^{1/2} \|BB^\dagger\|^{1/2}$ for any bounded A, B .

Let us prove (b). The statement follows from the Birman–Schwinger principle, see [1]. For completeness we sketch the proof here. By R7 for all $z > 0$ we have

$$H_0 + z^2 \geq (\epsilon + \lambda_n) V_{ik} \quad (45)$$

Forming a scalar product with $(H_0 + z^2)^{-1/2} \eta$, where $\eta \in D(H_0)$, $\|\eta\| = 1$ gives

$$(\epsilon + \lambda_n)(\eta, (H_0 + z^2)^{-1/2} V_{ik} (H_0 + z^2)^{-1/2} \eta) \leq 1 \quad (46)$$

This means that

$$\|\mathcal{C}_{ik;ik}(z)\| = \|(H_0 + z^2)^{-1/2} V_{ik} (H_0 + z^2)^{-1/2}\| \leq 1/(\epsilon + \lambda_n), \quad (47)$$

where we have used that $\|AA^\dagger\| = \|A^\dagger A\|$ for any bounded A . Thus (b) follows if we set $\delta = \epsilon/(\bar{\lambda} + \epsilon)$, where $\bar{\lambda} := \sup_n \lambda_n$. \square

We shall need the following

Lemma 8. *Suppose R1, R4-7 are satisfied. Then the operators*

$$\mathcal{R}_{ij}(\lambda_n) = [1 - \lambda_n \mathcal{C}_{ij;ij}(k_n)]^{-1} \quad \text{for } k_n = \sqrt{|E(\lambda_n)|} \quad (48)$$

are uniformly bounded for all n and converge in norm when $n \rightarrow \infty$.

Proof. By the previous Lemma the operators $\mathcal{C}_{ij;ij}(k_n)$ converge in norm to $\mathcal{C}_{ij;ij}(0)$ and $\lambda_n \|\mathcal{C}_{ij;ij}(k_n)\| < 1 - \delta$, where $\delta > 0$ is a constant. Now the result follows from expanding (48) in von Neumann series. \square

Lemma 9. *For $(ik) \neq (jm)$ the operator function $B_{ik}^{-1}(z)\mathcal{C}_{ik;jm}(z)$ is uniformly norm-bounded for $z \in (0, 1]$ and strongly continuous for $z \rightarrow +0$.*

Proof. We focus on $B_{12}^{-1}(z)\mathcal{C}_{12;23}(z)$, the other indices are treated similarly. Let us show that $\mathcal{F}_{12}B_{12}^{-1}(z)\mathcal{C}_{12;23}(z)\mathcal{F}_{12}^{-1}$ is uniformly bounded for $z \in (0, 1]$.

$$\mathcal{F}_{12}B_{12}^{-1}(z)\mathcal{C}_{12;23}(z)\mathcal{F}_{12}^{-1} = K_1(z) + K_2(z), \quad (49)$$

where

$$K_1(z) = \frac{1}{z+1} \mathcal{F}_{12}\mathcal{C}_{12;23}(z)\mathcal{F}_{12}^{-1}, \quad (50)$$

$$K_2(z) = \mathcal{F}_{12}\left(B_{12}^{-1}(z) - \frac{1}{z+1}\right)\mathcal{C}_{12;23}(z)\mathcal{F}_{12}^{-1}. \quad (51)$$

$K_1(z)$ is uniformly norm-bounded and for $z \rightarrow +0$ converges in norm by Lemma 7. Below we prove that the Hilbert-Schmidt norm of $K_2(z)$ is bounded for $z \in (0, 1]$. Let us first consider the Fourier transformed interaction term $\mathcal{F}_{12}V_{23}^{1/2}\mathcal{F}_{12}^{-1}$. In Jacobi coordinates the interaction term has the form $V_{23}^{1/2} = V_{23}^{1/2}(\beta x + \gamma y)$, where β and $\gamma \neq 0$ are real constants depending on masses $\beta = -m_2\hbar/((m_1+m_2)\sqrt{2\mu_{12}})$ and $\gamma = \hbar/\sqrt{2M_{12}}$. The Fourier transformed operator acts on $\phi(x, p_y)$ as

$$\mathcal{F}_{12}V_{23}^{1/2}\mathcal{F}_{12}^{-1}\phi = \frac{1}{(2\pi)^{3/2}\gamma^3} \int d^3p'_y \widehat{V_{23}^{1/2}}((p_y - p'_y)/\gamma) \exp\left\{i\frac{\beta}{\gamma}x \cdot (p_y - p'_y)\right\} \phi(x, p'_y), \quad (52)$$

where $\widehat{V_{23}^{1/2}} \in L^2(\mathbb{R}^3)$ is a Fourier transform of $V_{23}^{1/2} \in L^2(\mathbb{R}^3)$. For the kernel of $K_2(z)$ we get

$$\begin{aligned} K_2(x, p_y; x', p'_y) &= \frac{1}{2^{7/2}\pi^{5/2}\gamma^3} \left[\frac{1}{z+1+t(p_y)} - \frac{1}{z+1} \right] V_{12}^{1/2}(\alpha x) \\ &\times \frac{e^{-\sqrt{p_y^2+z^2}|x-x'|}}{|x-x'|} \exp\left\{i\frac{\beta}{\gamma}x' \cdot (p_y - p'_y)\right\} \widehat{V_{23}^{1/2}}((p_y - p'_y)/\gamma). \end{aligned} \quad (53)$$

For the square of the Hilbert-Schmidt norm we obtain

$$\|K_2(z)\|_2^2 = \frac{1}{2^7\pi^5} cc'\tilde{c} \int_{|p_y| \leq 1} d^3p_y \left[\frac{1}{z + \sqrt{|p_y|}} - \frac{1}{z+1} \right]^2 \frac{1}{\sqrt{p_y^2 + z^2}}, \quad (54)$$

where c, c' are defined in (30)–(31) and

$$\tilde{c} = \frac{1}{\gamma^6} \int d^3 p'_y |\widehat{V_{23}^{1/2}}(p'_y/\gamma)|^2 \quad (55)$$

is finite because $\widehat{V_{23}^{1/2}} \in L^2$. Estimating the integral in (54) we finally obtain

$$\|K_2(z)\|_2^2 \leq \frac{1}{2^7 \pi^5} c c' \tilde{c} \int_{|p_y| \leq 1} d^3 p_y \frac{1}{p_y^2} = \frac{1}{2^5 \pi^4} c c' \tilde{c}. \quad (56)$$

The strong continuity of $K_2(z)$ for $z \rightarrow +0$ follows from the explicit form of the kernel in (53). \square

Lemma 10. *Suppose $H(\lambda)$ defined in (14)–(15) satisfies R1, R4-7. If $\psi(\lambda_{n_k})$ is a weakly convergent subsequence of $\psi(\lambda_n)$, then $V_{ij}^{1/2} \psi(\lambda_{n_k})$ converges in norm.*

Proof. Let $J_s \in C^2(\mathbb{R}^{3N-3})$ denote the Ruelle–Simon partition of unity, see Definition 3.4 and Proposition 3.5 in [18]. For $s = 1, 2, 3$ one has $J_s \geq 0$, $\sum_s J_s^2 = 1$ and $J_s(\lambda x) = J_s(x)$ for $\lambda \geq 1$ and $|x| = 1$. Besides there exists $C > 0$ such that for $i \neq s$

$$\text{supp } J_s \cap \{x \mid |x| > 1\} \subset \{x \mid |r_i - r_s| \geq C|x|\}. \quad (57)$$

By the IMS formula (Theorem 3.2 in [18]) the Hamiltonian $H(\lambda)$ can be decomposed as

$$H(\lambda) = \sum_{s=1}^3 J_s H_s(\lambda) J_s + K(\lambda), \quad (58)$$

where

$$H_s(\lambda) = H_0 - \lambda V_{lm}, \quad (l \neq s, m \neq s) \quad (59)$$

$$K(\lambda) = -\lambda \sum_{s=1}^3 (V_{ls} + V_{ms}) |J_s|^2 + \sum_{s=1}^3 |\nabla J_s|^2 \quad (l \neq s, m \neq s, l \neq m). \quad (60)$$

By the properties of J_s one has $|\nabla J_s|^2 \in L^\infty(\mathbb{R}^{3N-3})$, which makes $|\nabla J_s|^2$ relatively H_0 -compact, see Lemma 7.11 in [15].

By condition of the lemma $\psi_k \xrightarrow{w} \psi_{cr}$, where $\psi_{cr} \in D(H_0)$ by Lemma 2 and for brevity we denote $\psi_k := \psi(\lambda_{n_k})$. We shall prove the lemma in three steps given by the following equations

$$(a) \quad \lim_{k \rightarrow \infty} \left((\psi_k - \psi_{cr}), K(\lambda_{n_k})(\psi_k - \psi_{cr}) \right) = 0 \quad (61)$$

$$(b) \quad \lim_{k \rightarrow \infty} \left((\psi_k - \psi_{cr}), H(\lambda_{n_k})(\psi_k - \psi_{cr}) \right) = 0 \quad (62)$$

$$(c) \quad \lim_{k \rightarrow \infty} \left((\psi_k - \psi_{cr}), V_{ij}(\psi_k - \psi_{cr}) \right) = 0. \quad (63)$$

From (c) the statement of the lemma clearly follows. Let us start with (a). From R6 we have

$$|(f, K(\lambda)f)| \leq (f, \tilde{K}f) \quad (\forall f \in D(H_0)), \quad (64)$$

where the operator \tilde{K} is defined through

$$\tilde{K} = \lambda \sum_{s=1}^3 (F_{ls} + F_{ms}) |J_s|^2 + \sum_{s=1}^3 |\nabla J_s|^2 \quad (l \neq s, m \neq s, l \neq m). \quad (65)$$

\tilde{K} is relatively H_0 -compact by Lemma 5 and thus by Lemma 2

$$((\psi_k - \psi_{cr}), \tilde{K}(\psi_k - \psi_{cr})) \rightarrow 0 \quad (66)$$

This proves (a). Rewriting the expression in (b) we obtain

$$\begin{aligned} ((\psi_k - \psi_{cr}), H(\lambda_{n_k})(\psi_k - \psi_{cr})) &= E(\lambda_{n_k})((\psi_k - \psi_{cr}), \psi_k) \\ &- ((\psi_k - \psi_{cr}), H(\lambda_{cr})\psi_{cr}) - [\lambda_{n_k} - \lambda_{cr}]((\psi_k - \psi_{cr}), V\psi_{cr}), \end{aligned} \quad (67)$$

where we have used $H(\lambda_{n_k}) = H(\lambda_{cr}) + [\lambda_{n_k} - \lambda_{cr}]V$. All terms on the rhs of (67) go to zero because $E(\lambda_{n_k}) \rightarrow 0$ and $\psi_k \xrightarrow{w} \psi_{cr}$. It remains to be shown that (c) is true.

$$\begin{aligned} \lim_{k \rightarrow \infty} ((\psi_k - \psi_{cr}), V_{ij}(\psi_k - \psi_{cr})) &= \sum_{s=1}^3 \lim_{k \rightarrow \infty} ((\psi_k - \psi_{cr}), J_s V_{ij} J_s (\psi_k - \psi_{cr})) \\ &= \lim_{k \rightarrow \infty} ((\psi_k - \psi_{cr}), J_l V_{ij} J_l (\psi_k - \psi_{cr})) \quad (l \neq i \neq j), \end{aligned} \quad (68)$$

where we have used that $J_i V_{ij}$ and $J_j V_{ij}$ are relatively H_0 -compact by Lemma 5 and the corresponding scalar products vanish by Lemma 2.

From (a), (b) and (58) we obtain

$$((\psi_k - \psi_{cr}), J_l H_l(\lambda_{n_k}) J_l (\psi_k - \psi_{cr})) \rightarrow 0 \quad (\forall l). \quad (69)$$

Together with R7 this gives us

$$\lim_{k \rightarrow \infty} ((\psi_k - \psi_{cr}), J_l V_{ij} J_l (\psi_k - \psi_{cr})) = 0 \quad (l \neq i \neq j). \quad (70)$$

Finally, comparing (70) and (68) we conclude that (c) holds. \square

Proof of Theorem 2. It is enough to show that any weakly converging subsequence of $\psi(\lambda_n)$ converges in norm. Indeed, in this case $\psi(\lambda_n)$ does not spread by Lemma 4 and thus by Theorem 1 there must exist a bound state at threshold. Suppose $\psi(\lambda_{n_s})$ is a weakly converging subsequence, that is $\psi(\lambda_{n_s}) \xrightarrow{w} \psi_{cr}$ and we must prove $\|\psi(\lambda_{n_s}) - \psi_{cr}\| \rightarrow 0$.

By Schrödinger equation for $k_{n_s}^2 = -E_{n_s} > 0$

$$\psi(\lambda_{n_s}) = \lambda_{n_s} \sum_{i < j} [H_0 + k_{n_s}^2]^{-1} V_{ij} \psi(\lambda_{n_s}) = \lambda_{n_s} \sum_{i < j} \mathcal{A}_{ij}(k_{n_s}) [B_{ij}^{-1}(k_{n_s}) V_{ij}^{1/2} \psi(\lambda_{n_s})], \quad (71)$$

where \mathcal{A}_{ij} is defined in (20). By Lemma 6 $\psi(\lambda_{n_s})$ converges in norm if the sequence $B_{ij}^{-1}(k_{n_s}) V_{ij}^{1/2} \psi(\lambda_{n_s})$ does. The convergence of the latter we prove below. From (71) we obtain

$$V_{ij}^{1/2} \psi(\lambda_{n_s}) = \lambda_{n_s} \sum_{l < m} \mathcal{C}_{ij;lm}(k_{n_s}) [V_{lm}^{1/2} \psi(\lambda_{n_s})]. \quad (72)$$

Using (48) we rewrite (72)

$$V_{ij}^{1/2}\psi(\lambda_{n_s}) = \lambda_{n_s}\mathcal{R}_{ij}(k_{n_s}) \sum_{l < m(lm) \neq (ij)} \mathcal{C}_{ij;lm}(k_{n_s})(V_{lm}^{1/2}\psi(\lambda_{n_s})). \quad (73)$$

Now we act with $B_{ij}^{-1}(k_{n_s})$ on both parts of (73) and use that it commutes with $\mathcal{R}_{ij}(k_{n_s})$

$$B_{ij}^{-1}(k_{n_s})V_{ij}^{1/2}\psi(\lambda_{n_s}) = \lambda_{n_s}\mathcal{R}_{ij}(k_{n_s}) \sum_{l < m(lm) \neq (ij)} B_{ij}^{-1}(k_{n_s})\mathcal{C}_{ij;lm}(k_{n_s})(V_{lm}^{1/2}\psi(\lambda_{n_s})). \quad (74)$$

By Lemmas 8,9,10 the rhs converges in norm. \square

In the next sections our aim is to analyse the case when one pair of particles has a zero-energy resonance. We would show (Theorem 3) that in this case Theorem 2 does not generally hold. This shows that the condition of Theorem 2 on the absence of resonances in particle pairs is essential.

4. A Zero Energy Resonance in a 2-Particle System

In this section we shall use the method of [1] to prove a result similar to Lemma 2.2 in [8]. Let us consider the Hamiltonian of 2 particles in \mathbb{R}^3

$$h_{12}(\varepsilon) := -\Delta_x - (1 + \varepsilon)V_{12}(\alpha x), \quad (75)$$

where $\varepsilon \geq 0$ is a parameter and α is defined right after (25). Additionally, we require

$\overline{R1}$ $0 \leq V_{12}(\alpha x) \leq b_1 e^{-b_2|x|}$, where $b_{1,2} > 0$ are constants.

$\overline{R2}$ $h_{12}(0) \geq 0$ and $\sigma(h_{12}(\varepsilon)) \cap (-\infty, 0) \neq \emptyset$ for $\varepsilon > 0$.

The requirement $\overline{R2}$ means that $h_{12}(0)$ has a resonance at zero energy, that is, negative energy bound states emerge iff the coupling constant is incremented by an arbitrary amount (in terminology of [1] the system is at the coupling constant threshold).

The following integral operator appears in the Birman-Schwinger principle [14, 1]

$$L(k) := \sqrt{V_{12}} \left(-\Delta_x + k^2 \right)^{-1} \sqrt{V_{12}}. \quad (76)$$

$L(k)$ is analytic for $\operatorname{Re} k > 0$. Due to $\overline{R1}$ one can use the integral representation and analytically continue $L(k)$ into the interior of the disk on the complex plane, which has its centre at $k = 0$ and the radius $|b_2|$ [1]. The analytic continuation is denoted as $\tilde{L}(k) = \sum_n \tilde{L}_n k^n$, where \tilde{L}_n are Hilbert-Schmidt operators.

Remark. In Sec. 2 in [1] (page 255) Klaus and Simon consider only finite range potentials. In this case $L(k)$ can be analytically continued into the whole complex plane. As the authors mention it in Sec. 9 the case of potentials with an exponential fall off requires only a minor change: $L(k)$ extends analytically as a bounded operator to the domain $\{k \mid \operatorname{Re} k > -b_2\}$.

Under requirements $\overline{R}1$, $\overline{R}2$ the operator $L(0) = \tilde{L}(0)$ is Hilbert-Schmidt and its maximal eigenvalue is equal to one

$$L(0)\phi_0 = \phi_0. \quad (77)$$

$L(0)$ is positivity-preserving, hence, the maximal eigenvalue is non-degenerate and $\phi_0 \geq 0$. We choose the normalization, where $\|\phi_0\| = 1$.

By the standard Kato-Rellich perturbation theory [19, 14] there exists $\rho > 0$ such that for $|k| \leq \rho$

$$\tilde{L}(k)\phi(k) = \mu(k)\phi(k), \quad (78)$$

where $\mu(k), \phi(k)$ are analytic, $\mu(0) = 1$, $\phi(0) = \phi_0$ and the eigenvalue $\mu(k)$ is non-degenerate. By Theorem 2.2 in [1]

$$\mu(k) = 1 - ak + O(k^2), \quad (79)$$

where

$$a = (\phi_0, (V_{12})^{1/2})^2 / (4\pi) > 0. \quad (80)$$

The orthonormal projection operators

$$\mathbb{P}(k) := (\phi(k), \cdot)\phi(k) = (\phi_0, \cdot)\phi_0 + \mathcal{O}(k), \quad (81)$$

$$\mathbb{Q}(k) := 1 - \mathbb{P}(k) \quad (82)$$

are analytic for $|k| < \rho$ as well. Our aim is to analyse the following operator function on $k \in (0, \infty)$

$$W(k) = [1 - L(k)]^{-1}. \quad (83)$$

By the Birman-Schwinger principle $\|L(k)\| < 1$ for $k > 0$, which makes $W(k)$ well-defined.

Lemma 11. *There exists $0 < \rho_0 < 1$ such that for $k \in (0, \rho_0)$*

$$W(k) = \frac{\mathbb{P}_0}{ak} + \mathcal{Z}(k), \quad (84)$$

where $\mathbb{P}_0 := (\phi_0, \cdot)\phi_0$ and $\sup_{k \in (0, \rho_0)} \|\mathcal{Z}(k)\| < \infty$.

Proof. $\tilde{L}(k) = L(k)$ when $k \in (0, \rho)$. We get from (83)

$$\begin{aligned} W(k) &= [1 - L(k)]^{-1} = [1 - L(k)]^{-1}\mathbb{P}(k) + [1 - L(k)]^{-1}\mathbb{Q}(k) \\ &= [1 - \mu(k)\mathbb{P}(k)]^{-1}\mathbb{P}(k) + [1 - \mathbb{Q}(k)L(k)]^{-1}\mathbb{Q}(k) \\ &= \frac{1}{1 - \mu(k)}\mathbb{P}(k) + \mathcal{Z}'(k), \end{aligned} \quad (85)$$

where

$$\mathcal{Z}'(k) := [1 - \mathbb{Q}(k)L(k)]^{-1}\mathbb{Q}(k). \quad (86)$$

Note that $\sup_{k \in (0, \rho)} \|\mathbb{Q}(k)L(k)\| < 1$ because the eigenvalue $\mu(k)$ remains isolated for $k \in [0, \rho)$. Thus $\mathcal{Z}'(k) = \mathcal{O}(1)$. Using (79), (80) and (81) proves the lemma. Clearly, one can always choose $\rho_0 < 1$. \square

Remark. The singularity of $W(k)$ near $k = 0$ has been analysed in [8] (Lemma 2.2 in [8]), see also [9]). The decomposition (84) differs in the sense that $\mathcal{Z}(k)$ is uniformly bounded in the vicinity of $k = 0$. The price we paid for it is the requirement $\overline{R}1$ on the exponential fall off of V_{12} .

5. Zero Energy Resonance in a 3-Particle system

Let us consider the Schrödinger operator for three particles in \mathbb{R}^3

$$H = H_0 - V_{12}(r_1 - r_2) - V_{13}(r_1 - r_3) - V_{23}(r_2 - r_3), \quad (87)$$

where r_i are particle position vectors and H_0 is the kinetic energy operator with the centre of mass removed. Apart from $\overline{R}1$, $\overline{R}2$ we shall need the following additional requirement

$$\overline{R}3 \quad V_{13}, V_{23} \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3) \text{ and } V_{13}, V_{23} \geq 0 \text{ and } V_{23} \neq 0.$$

Here we shall prove

Theorem 3. *Suppose H defined in (87) satisfies $\overline{R}1$, $\overline{R}2$, $\overline{R}3$. Suppose additionally that $H \geq 0$ and $H\psi_0 = 0$, where $\psi_0 \in D(H_0)$. Then $\psi_0 = 0$.*

We defer the proof to the end of this section. Our next aim is to derive the inequality (103)-(104).

We use the same Jacobi coordinates as in Sec. 3 so that (16) holds. The full set of coordinates in \mathbb{R}^6 is labelled by ξ . We shall need the following trivial technical lemmas.

Lemma 12. *Suppose an operator A is positivity preserving and $\|A\| < 1$. Then $(1-A)^{-1}$ is bounded and positivity preserving.*

Proof. A simple expansion of $(1-A)^{-1}$ into von Neumann series. □

Lemma 13. *Suppose $g \in L^1(\mathbb{R}^3)$, $\|g\|_1 > 0$ and $g(y) \geq 0$. Then for all $\epsilon_0 > 0$*

$$\lim_{z \rightarrow +0} \int_{|p_y| \leq \epsilon_0} d^3 p_y \frac{|\hat{g}|^2}{(p_y^2 + z^2)^{3/2}} = \infty \quad (88)$$

Proof. Let us set

$$J_\epsilon(z) = \int_{|p_y| \leq \epsilon} d^3 p_y \frac{1}{(p_y^2 + z^2)^{3/2}} \left| \int d^3 y e^{ip_y \cdot y} g(y) \right|^2. \quad (89)$$

We have

$$J_\epsilon(z) \geq \int_{|p_y| \leq \epsilon} d^3 p_y \frac{1}{(p_y^2 + z^2)^{3/2}} \left| \int d^3 y g(y) \cos(p_y \cdot y) \right|^2. \quad (90)$$

Let us fix r so that

$$\int_{|y| > r} d^3 y g(y) = \frac{1}{4} \|g\|_1 \quad (91)$$

Setting $\epsilon = \min[\epsilon_0, \pi/(3r)]$ we get

$$\cos(p_y \cdot y) \geq \frac{1}{2} \quad \text{if} \quad |p_y| \leq \epsilon, |y| \leq r. \quad (92)$$

Substituting (92) and (91) into (90) we get

$$J_{\epsilon_0}(z) \geq J_\epsilon(z) \geq \frac{\|g\|_1^2}{64} \int_{|p_y| \leq \epsilon} d^3 p_y \frac{1}{(p_y^2 + z^2)^{3/2}}. \quad (93)$$

The integral in (93) logarithmically diverges for $z \rightarrow +0$. \square

So let us assume that there is a bound state $\psi_0 \in D(H_0)$ at zero energy, where $\psi_0 > 0$ because it is the ground state, see [14] Sec. XIII.12. Then we would have

$$H_0 \psi_0 = V_{12} \psi_0 + V_{13} \psi_0 + V_{23} \psi_0, \quad (94)$$

Adding the term $z^2 \psi_0$ (where here and further $z > 0$) and acting with an inverse operator on both sides of (94) gives

$$\begin{aligned} \psi_0 &= [H_0 + z^2]^{-1} V_{12} \psi_0 + [H_0 + z^2]^{-1} V_{13} \psi_0 + [H_0 + z^2]^{-1} V_{23} \psi_0 \\ &\quad + z^2 [H_0 + z^2]^{-1} \psi_0. \end{aligned} \quad (95)$$

From now we let z vary in the interval $(0, \rho_0/2)$, where $\rho_0 < 1$ was defined in Lemma 11. The operator $[H_0 + z^2]^{-1}$ is positivity preserving, see, for example, [14], Example 3 from Sec. IX.7 in vol. 2 and Theorem XIII.44 in vol. 4. Thus we obtain the inequality

$$\psi_0 \geq [H_0 + z^2]^{-1} \sqrt{V_{12}} (\sqrt{V_{12}} \psi_0) \quad (96)$$

Now let us focus on the term $\sqrt{V_{12}} \psi_0$. Using (95) we get

$$\begin{aligned} \left[1 - \sqrt{V_{12}} (H_0 + z^2)^{-1} \sqrt{V_{12}}\right] \sqrt{V_{12}} \psi_0 &= \sqrt{V_{12}} [H_0 + z^2]^{-1} V_{13} \psi_0 \\ &\quad + \sqrt{V_{12}} [H_0 + z^2]^{-1} V_{23} \psi_0 + z^2 \sqrt{V_{12}} [H_0 + z^2]^{-1} \psi_0 \end{aligned} \quad (97)$$

And by Lemma 12

$$\sqrt{V_{12}} \psi_0 \geq \left[1 - \sqrt{V_{12}} (H_0 + z^2)^{-1} \sqrt{V_{12}}\right]^{-1} \sqrt{V_{12}} [H_0 + z^2]^{-1} V_{23} \psi_0. \quad (98)$$

The resolvent identity reads

$$[H_0 + z^2]^{-1} - [H_0 + 1]^{-1} = (1 - z^2) [H_0 + 1]^{-1} [H_0 + z^2]^{-1}. \quad (99)$$

Clearly, for $z \in (0, 1)$ the difference on the lhs of (99) is a positivity preserving operator. Using this fact we can transform (98) into

$$\sqrt{V_{12}} \psi_0 \geq \left[1 - \sqrt{V_{12}} (H_0 + z^2)^{-1} \sqrt{V_{12}}\right]^{-1} \sqrt{V_{12}} [H_0 + 1]^{-1} V_{23} \psi_0. \quad (100)$$

It is technically convenient to cut off the wave function ψ_0 by introducing

$$\psi_1(\xi) := \psi_0(\xi) \chi_{\{|\xi| \leq b\}}, \quad (101)$$

where clearly $\psi_1 \in L^2 \cap L^1(\mathbb{R}^6)$ and $b > 0$ is fixed so that $\|V_{23} \psi_1\| \neq 0$ (which is always possible since $V_{23} \neq 0$).

Applying again Lemma 12 we get out of (100)

$$\sqrt{V_{12}}\psi_0 \geq \left[1 - \sqrt{V_{12}}(H_0 + z^2)^{-1}\sqrt{V_{12}}\right]^{-1} \sqrt{V_{12}}[H_0 + 1]^{-1}V_{23}\psi_1. \quad (102)$$

Substituting (102) into (96) gives that for all $z \in (0, \rho_0/2)$

$$\psi_0 \geq f(z) \geq 0, \quad (103)$$

where

$$\begin{aligned} f(z) &= [H_0 + z^2]^{-1} \sqrt{V_{12}} \left[1 - \sqrt{V_{12}}(H_0 + z^2)^{-1} \sqrt{V_{12}}\right]^{-1} \\ &\times \sqrt{V_{12}}[H_0 + 1]^{-1} V_{23} \psi_1. \end{aligned} \quad (104)$$

Our aim is to prove that $\lim_{z \rightarrow +0} \|f(z)\| = \infty$, which would be in contradiction with (103) because ψ_0 is the normalized ground state wave function. Let us define

$$\Phi(x, y) := [H_0 + 1]^{-1} V_{23} \psi_1, \quad (105)$$

$$g(y) := \int dx \Phi(x, y) \sqrt{V_{12}(\alpha x)} \phi_0(x), \quad (106)$$

where ϕ_0 is defined in (77).

Lemma 14. $g \in L^1 \cap L^2(\mathbb{R}^3)$ and $\|g\|_1 > 0$.

Proof. Following [1] let us denote by $G_0(\xi - \xi', 1)$ the integral kernel of $[H_0 + 1]^{-1}$. We need a rough upper bound on $G_0(\xi, 1)$. Using the formula on p. 262 in [1] we get

$$\begin{aligned} (4\pi)^3 |\xi|^4 e^{|\xi|/2} G_0(\xi, 1) &= \int_0^\infty t^{-3} e^{|\xi|/2} e^{-t|\xi|^2} e^{-1/(4t)} dt \\ &\leq \int_0^\infty t^{-3} e^{-3/(16t)} dt = \frac{256}{9} \end{aligned} \quad (107)$$

Hence,

$$G_0(\xi, 1) \leq \frac{4}{9\pi|\xi|^4} e^{-|\xi|/2}. \quad (108)$$

Using $\|\sqrt{V_{12}}\phi_0\|_\infty < \infty$ we get $g \in L^1 \cap L^2(\mathbb{R}^3)$ if $\Phi \in L^1 \cap L^2(\mathbb{R}^6)$. Because $\Phi \in L^2(\mathbb{R}^6)$ to prove $\Phi \in L^1(\mathbb{R}^6)$ it suffices to show that $\chi_{\{|\xi| \geq 2b\}} \Phi(\xi) \in L^1(\mathbb{R}^6)$, where b was defined after Eq. (101). This follows from (108)

$$\begin{aligned} \chi_{\{|\xi| \geq 2b\}} \Phi(\xi) &\leq \chi_{\{|\xi| \geq 2b\}} \int_{|\xi'| \leq b} d^6 \xi' G_0(\xi - \xi', 1) V_{23} \psi_1(\xi') \\ &\leq \chi_{\{|\xi| \geq 2b\}} \frac{4}{9\pi(|\xi| - b)^4} e^{-(|\xi| - b)/2} \|V_{23} \psi_1\|_1 \in L^1(\mathbb{R}^6) \end{aligned} \quad (109)$$

From $\Phi(x, y) > 0$ it follows that $\|g\|_1 > 0$. □

Applying \mathcal{F}_{12} to (104) we get

$$\begin{aligned} \hat{f}(z) &= [-\Delta_x + p_y^2 + z^2]^{-1} \sqrt{V_{12}} \left[1 - \sqrt{V_{12}}(-\Delta_x + p_y^2 + z^2)^{-1} \sqrt{V_{12}}\right]^{-1} \\ &\times \sqrt{V_{12}}[-\Delta_x + p_y^2 + 1]^{-1} \widehat{V_{23} \psi_1}. \end{aligned} \quad (110)$$

From now on $z \in (0, \rho_0/2)$. By Lemma 11 for $|p_y| < \rho_0/2$ and $z < \rho_0/2$

$$\left[1 - \sqrt{V_{12}}(-\Delta_x + p_y^2 + z^2)^{-1} \sqrt{V_{12}}\right]^{-1} = \frac{\mathbb{P}_0}{a\sqrt{p_y^2 + z^2}} + \mathcal{Z}\left(\sqrt{p_y^2 + z^2}\right), \quad (111)$$

where a and $\phi_0(x)$ are defined in Sec. 4 and \mathbb{P}_0 acts on $u(x, p_y)$ as $\mathbb{P}_0 u(x, p_y) = \phi_0(x) \int \phi_0(x') u(x', p_y) dx'$. Substituting (111) into (110) and denoting for brevity $\chi_0(p_y) := \chi_{\{|p_y| < \rho_0/2\}}$ we obtain

$$\chi_0(p_y) \hat{f}(z) = \hat{f}_1(z) + \hat{f}_2(z), \quad (112)$$

where

$$\hat{f}_1(z) = \chi_0(p_y) \frac{\hat{g}(p_y)}{\sqrt{p_y^2 + z^2}} [-\Delta_x + p_y^2 + z^2]^{-1} (\sqrt{V_{12}} \phi_0(x)), \quad (113)$$

$$\begin{aligned} \hat{f}_2(z) &= \chi_0(p_y) [-\Delta_x + p_y^2 + z^2]^{-1} \sqrt{V_{12}} \mathcal{Z}\left(\sqrt{p_y^2 + z^2}\right) \\ &\quad \sqrt{V_{12}} [-\Delta_x + p_y^2 + 1]^{-1} (\mathcal{F}_{12} V_{23} \mathcal{F}_{12}^{-1}) \hat{\psi}_1, \end{aligned} \quad (114)$$

and we have used (105)-(106). The next lemma follows from the results of Sec. 3.

Lemma 15. $\sup_{z \in (0, \rho_0/2)} \|f_2(z)\| < \infty$

Proof. Let us rewrite (114) in the form

$$f_2(z) = \mathcal{A}(z) \mathcal{B}(z) \mathcal{C}(z) \psi_0, \quad (115)$$

where

$$\mathcal{A}(z) = \chi_0(p_y) [-\Delta_x + p_y^2 + z^2]^{-1} \sqrt{V_{12}} [1 + t(p_y) + z], \quad (116)$$

$$\mathcal{B}(z) = \chi_0(p_y) \mathcal{Z}\left(\sqrt{p_y^2 + z^2}\right), \quad (117)$$

$$\mathcal{C}(z) = \chi_0(p_y) \sqrt{V_{12}} [-\Delta_x + p_y^2 + 1]^{-1} [1 + t(p_y) + z]^{-1} (\mathcal{F}_{12} V_{23} \mathcal{F}_{12}^{-1}), \quad (118)$$

and $t(p_y)$ is defined as in (19). Note that by (84) $\mathcal{Z}\left(\sqrt{p_y^2 + z^2}\right)$ is a difference of two operators each of which commutes with the operator of multiplication by $[1 + t(p_y) + z]$. We need to show that each of the three operators in the product in (115) are uniformly norm-bounded for $z \in (0, \rho_0/2)$. That $\sup_{z \in (0, \rho_0/2)} \|\mathcal{B}(z)\| < \infty$ follows from Lemma 11. That $\sup_{z \in (0, \rho_0/2)} \|\mathcal{A}(z)\|, \|\mathcal{C}(z)\| < \infty$ follows from the proofs of Lemmas 6, 9 in Sec. 3. Let us, however, repeat the argument here. Taking into account that $0 < z < \rho_0/2 < 1$ we obtain

$$\begin{aligned} \|\mathcal{A}(z)\| &= \left\| \chi_0(p_y) [-\Delta_x + p_y^2 + z^2]^{-1} \sqrt{V_{12}} [1 + t(p_y) + z] \right\| \\ &\leq \left\| \chi_0(p_y) [-\Delta_x + p_y^2 + z^2]^{-1} \sqrt{V_{12}} \sqrt{|p_y|} \right\| + z \left\| \chi_0(p_y) [-\Delta_x + p_y^2 + z^2]^{-1} \sqrt{V_{12}} \right\| \\ &\leq \left\| \chi_0(p_y) [-\Delta_x + p_y^2 + z^2]^{-1} \sqrt{V_{12}} \sqrt{|p_y|} \right\| + z \left\| [-\Delta_x + z^2]^{-1} \sqrt{V_{12}} \right\| \end{aligned} \quad (119)$$

It is trivial to estimate the squares of the norms on the rhs if one uses the explicit expressions for the operator kernels. For example,

$$\begin{aligned} & \left\| \chi_0(p_y) [-\Delta_x + p_y^2 + z^2]^{-1} \sqrt{V_{12}} \sqrt{|p_y|} \right\|^2 \\ & \leq \frac{1}{(4\pi)^2} \sup_{|p_y| < \rho_0/2} |p_y| \int \int \frac{e^{-2|p_y||x-x'|} V_{12}(\alpha x')}{|x-x'|^2} d^3x d^3x' = \frac{cc'}{4\pi} < \infty, \end{aligned} \quad (120)$$

where c, c' are defined in (30)–(31). The second norm in (119) is estimated similarly and the result is that $\mathcal{A}(z)$ is uniformly norm-bounded for $z \in (0, \rho_0/2)$. Using (52) we can write the integral kernel of $\mathcal{C}(z)$ as

$$\begin{aligned} \mathcal{C}(z)(x, p_y; x', p'_y) &= \frac{\chi_0(p_y)}{2^{7/2} \pi^{5/2} \gamma^3} \left[z + \sqrt{|p_y|} \right]^{-1} V_{12}^{1/2}(\alpha x) \\ &\times \frac{e^{-\sqrt{p_y^2 + z^2}|x-x'|}}{|x-x'|} \exp \left\{ i \frac{\beta}{\gamma} x' \cdot (p_y - p'_y) \right\} \widehat{V_{23}^{1/2}}((p_y - p'_y)/\gamma). \end{aligned} \quad (121)$$

Estimating $\|\mathcal{C}(z)\|^2$ through the square of the Hilbert–Schmidt norm results in

$$\|\mathcal{C}(z)\|^2 \leq \frac{cc'\tilde{c}}{2^7 \pi^5} \int_{|p_y| \leq \rho_0/2} \frac{d^3 p_y}{|p_y|(z + \sqrt{|p_y|})^2}, \quad (122)$$

where \tilde{c} is defined in (55). From (122) it follows that $\sup_{z \in (0, \rho_0/2)} \|\mathcal{C}(z)\| < \infty$. \square

The last Lemma needed for the proof of Theorem 3 is

Lemma 16. $\lim_{z \rightarrow 0} \|f_1(z)\| = \infty$.

Proof. We get

$$\begin{aligned} \|\hat{f}_1(z)\|^2 &= \frac{1}{4\pi^2} \int_{|p_y| \leq \rho_0/2} dp_y \frac{|\hat{g}(p_y)|^2}{p_y^2 + z^2} \int dx \int dx' \int dx'' \frac{e^{-\sqrt{p_y^2 + z^2}|x-x'|}}{|x-x'|} \\ &\times \frac{e^{-\sqrt{p_y^2 + z^2}|x-x''|}}{|x-x''|} (\sqrt{V_{12}}(\alpha x') \phi_0(x')) (\sqrt{V_{12}}(\alpha x'') \phi_0(x'')). \end{aligned} \quad (123)$$

There are constants $R_0, C_0 > 0$ such that

$$\int d^3x' \frac{e^{-\delta|x-x'|}}{|x-x'|} \sqrt{V_{12}(\alpha x')} \phi_0(x') \geq C_0 \frac{e^{-2\delta|x|}}{|x|} \chi_{\{|x| \geq R_0\}} \quad (124)$$

for all $\delta > 0$. Indeed, the following inequality holds for all $R_0 > 0$

$$\chi_{\{|x| \geq R_0\}} \frac{e^{-\delta|x-x'|}}{|x-x'|} \chi_{\{|x'| \leq R_0\}} \geq \frac{e^{-2\delta|x|}}{2|x|} \chi_{\{|x| \geq R_0\}}. \quad (125)$$

Substituting (125) into the lhs of (124) we obtain (124), where

$$C_0 = \frac{1}{2} \int_{|x'| \leq R_0} d^3x' \sqrt{V_{12}(\alpha x')} \phi_0(x') \quad (126)$$

and one can always choose R_0 so that $C_0 > 0$. Using (124) we get

$$\|\hat{f}_1(z)\|^2 \geq c \int_{|p_y| \leq \frac{\rho_0}{2}} dp_y \frac{|\hat{g}(p_y)|^2}{(p_y^2 + z^2)^{3/2}}, \quad (127)$$

where $c > 0$ is a constant. Now the result follows from Lemmas 13, 14. \square

The proof of Theorem 3 is now trivial.

Proof of Theorem 3. A bound state at threshold should it exist must satisfy inequality (103) for all $z \in (0, \rho_0/2)$. Thus $\|f(z)\|$ and, hence, $\|\chi_0 \hat{f}(z)\|$ are uniformly bounded for $z \in (0, \rho_0/2)$. By (112) and Lemmas 15, 16 this leads to a contradiction. \square

6. Example of a Three-Particle Zero Energy Resonance and Physical Applications

Suppose that $\overline{R}2$ is fulfilled. Let us rewrite (87) using additional coupling constants $\Theta, \Lambda > 0$

$$H(\Theta, \Lambda) = [-\Delta_x - V_{12}] - \Delta_y - \Theta V_{13} - \Lambda V_{23}. \quad (128)$$

For simplicity, let us require that $V_{ik} \geq 0$ and $V_{ik} \in C_0^\infty(\mathbb{R}^3)$. Let $\Theta_{cr}, \Lambda_{cr}$ denote the 2-particle coupling constant thresholds for particle pairs 1,3 and 2,3 respectively. On one hand, using a variational argument it is easy to show that there exists $\epsilon > 0$ such that $H(\Theta, \Lambda) > 0$ if $\Theta, \Lambda \in [0, \epsilon]$ (that is in this range $H(\Theta, \Lambda)$ has neither negative energy bound states nor a zero energy resonance) [20, 21]. On the other hand, from the Yafaev's rigorous proof of the existence of the Efimov effect [9] we know that $H(\Theta_{cr}, \Lambda)$ has an infinite number of negative energy bound states for $\Lambda \in [0, \Lambda_{cr})$ because in this case two of the binary subsystems have zero energy resonances. So let us fix $\Lambda = \epsilon$ and let Θ vary in the range $[\epsilon, \Theta_{cr}]$. The energy of the ground state $E_{gr}(\Theta) = \inf \sigma(H(\Theta, \epsilon))$ is a continuous function of Θ . $E_{gr}(\Theta)$ decreases monotonically at the points where $E_{gr}(\Theta) < 0$. Because $E_{gr}(\epsilon) = 0$ there must exist $\Theta_0 \in (\epsilon, \Theta_{cr})$ such that $E_{gr}(\Theta) < 0$ for $\Theta \in (\Theta_0, \Theta_{cr})$ and $E_{gr}(\Theta_0) = 0$.

Summarizing, $H(\Theta_0, \epsilon)$ is at the 3-particle coupling constant threshold. By Theorem 3 $H(\Theta_0, \epsilon)$ has a zero energy resonance but not a zero energy bound state. If $\psi_{gr}(\Theta, \xi) \in L^2(\mathbb{R}^6)$ is a wave function of the ground state defined on the interval (Θ_0, Θ_{cr}) then for $\Theta \rightarrow \Theta_0 + 0$ the wave function must totally spread (see Sec. 2). Which means that for any $R > 0$

$$\lim_{\Theta \rightarrow \Theta_0 + 0} \int_{|\xi| < R} |\psi_{gr}(\Theta, \xi)|^2 d\xi \rightarrow 0. \quad (129)$$

Remember also [2] that if the particles 1 and 2 would be bound with the energy $e_{12} < 0$ then the 3-particle system cannot have a square integrable ground state wave function at the energy e_{12} . Though in the paper we restricted our analysis to the case of non-positive pair interactions, with additional effort one can show that the main results also hold without this restriction.

In physics there is now an increased interest to the systems, which exhibit unusually large spatial extension and form the so-called *halo*. Under halo one usually means [11] that the substantial part of the wave function is located in the classically forbidden region so that some interparticle distances exceed by far the range of the interaction.

The interest in such systems started with the study of light atomic nuclei but the concept has now penetrated atomic and molecular physics [11]. A typical example of halo systems are weakly bound nuclei ${}^6\text{He}$ and ${}^{11}\text{Li}$, where one finds a pronounced three-particle structure consisting of a tightly bound cluster (${}^4\text{He}$ and ${}^9\text{Li}$ respectively) and two neutrons. These nuclei treated as a three-particle system form the so-called Borromean structure (the term originates from the Italian heraldic), which means that if one of the particles are removed, the remaining two fall apart. The two neutrons orbiting around the core form a halo and the effective size of these systems is by far larger than that of normal stable nuclei having nearly the same mass.

Numerical calculations [10] showed that a deeply lying resonance in the two-neutron interaction is important in reproducing the neutron halo. Even such a “naive” model, where two neutrons are treated as a bound particle called *dineutron* [10] is still effectively being used today. The extensive three-body calculations often approximate the neutron-neutron interaction by a simple Gaussian, where the parameters are tuned so as to accommodate a low lying resonance. In the physics literature (see, for example [10, 11]) one often uses the asymptotic of the bound state wave function due to Merkuriev [22] $\psi \simeq \rho^{-5/2}e^{-k\rho}$, where ρ is the hyperradius and k is proportional to square root of the binding energy. Here one should be warned against relying on the validity of this asymptotic behaviour near the threshold. The results obtained here show that this can be misleading. Indeed, the normalized sequence of functions $c_n\rho^{-5/2}e^{-k_n\rho}$, where $c_n := \|\rho^{-5/2}e^{-k_n\rho}\|^{-1}$ totally spreads in the limit of vanishing binding $k_n \rightarrow 0$. As we know now the wave function would not totally spread unless one pair of particles would have a zero energy resonance. It is worth mentioning that exactly at the zero energy threshold the wave function does not have an exponential fall off. From the Green’s function bound [2] it follows that $\psi_{gr} \geq \rho^{-4}$, where ψ_{gr} is the normalized ground state at zero energy threshold, which can be chosen positive. The results presented here contribute to setting the general theory of halos on a rigorous footing.

Another example of spatially extended Borromean structures are the so-called Efimov states. The Efimov states predicted by V. Efimov [23] attracted considerable interest due to their bizarre and counter-intuitive properties. These states start to appear when at least two of the binary subsystems either have very large scattering lengths or bound states at nearly zero energy. In the limiting case when at least two of the binary subsystems have zero energy resonances the number of such states is infinite. In that case the binding energy of the n -th state decreases exponentially with n , and bound states attain enormous spatial extension. (The infinite sequence of Efimov states ψ_n totally spreads because $\psi_n \xrightarrow{w} 0$ due to orthogonality of the states and $\sup_n \|H_0\psi_n\| < \infty$, see Lemmas 1, 3). The existence of this effect was demonstrated rigorously by Yafaev in [9], see also [8]. Remarkably, the three-particle system has an infinite number of bound states in spite of the fact that all its subsystems are unbound. These states evaded any experimental evidence for 35 years since their prediction until Kraemer *et al.* [12] reported on their discovery in an ultracold gas of cesium atoms. The present paper predicts extended halo-like states for three atoms near zero energy

threshold, if one pair of atoms has a large scattering length (that is it is close to the zero energy resonance). Therefore, we advise experimentalists and theoreticians to look for such states in ultracold gas mixtures prepared through the appropriate Feshbach resonance tuning [24]. In analogy with Efimov states they can be, probably, indirectly detected through the 3-body recombination loss.

Acknowledgments

The author would like to thank Prof. Walter Greiner for the warm hospitality at FIAS.

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